Spaces of Bounded Funcitons

BIO 251,
Lab 4

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Space of Bounded Functions

- Let T be an arbitrary set.
- The space $\ell^\infty(T)$ is defined as the set of all uniformly bounded real functions $z : T \mapsto \mathbb{R}$ with:

$$||z||_T := \sup_{t \in T} |z(t)| < \infty$$

Stochastic Process

Stochastic Process

Definition. A *stochastic process* is an indexed collection of random variables $\{X(t) : t \in T\}$ defined on the same probability space:

- i.e. every $X(t) : \Omega \mapsto \mathbb{R}$ is a measurable map.
- If the *sample paths* $t \mapsto X(t, \omega)$ are bounded, then a stochastic process yields a map $X : \Omega \mapsto \ell^\infty(T)$.

Two Lemmas

Lemma 1.

Let $X_n : \Omega_n \mapsto \ell^\infty(T)$ be asymptotically tight. Then it is asymptotically measurable iff $X_n(t)$ is asymptotically measurable for every $t \in T$.

Lemma 2.

Let X and Y be tight Borel measurable maps into $\ell^\infty(T)$. Then X and Y are equal in Borel law iff all corresponding marginals of X and Y are equal in law.

And a Theorem

Theorem.

(i) Let $X_n : \Omega_n \mapsto \ell^{\infty}(T)$ be arbitrary. Then X_n converges weakly to a tight limit iff X_n is asymptotically tight, and the marginals $(X_n(t_1), \ldots, X_n(t_k))$ converge weakly to a limit for every finite subset $t_1, \ldots, t_k$ of T.

(ii) If X_n is asymptotically tight and it's marginals converge weakly to the marginals $(X(t_1), \ldots, X(t_k))$ of a stochastic process X then there is a version of X with uniformly bounded sample paths and $X_n \rightsquigarrow X$.

Characterization of Asymptotic Tightness

Theorem.

A sequence $X_n : \Omega_n \mapsto \ell^\infty(T)$ is asymptotically tight, if and only if $X_n(t)$ is asymptotically tight in $\mathbb{R}$ for every t, and there exists a semimetric ρ on T such that (T, ρ) is totally bounded and X_n is *asymptotically uniformly ρ-equicontinuous in probability*, i.e. for every $\varepsilon, \eta > 0$, there exist a $\delta > 0$ such that:

$$\limsup_n P^* \left(\sup_{\rho(s,t)<\delta} |X_n(s) - X_n(t)| > \varepsilon \right) < \eta$$

Characterization of Asymptotic Tightness

Addendum.

If, moreover, $X_n \rightsquigarrow X$, then almost all paths $t \mapsto X(t, \omega)$ are uniformly ρ-continuous; and the semimetric ρ can WLOG be taken equal to any semimetric ρ for which this is true and (T, ρ) is totally bounded.

Examples of Semimetrics

- $\rho_0(s,t) = \mathrm{E} \arctan |X(s) - X(t)|$
- $\rho_p(s,t) = [\mathrm{E}\,|X(s) - X(t)|^p]^{1/(p \vee 1)}$

Gaussian Processes

Definition (Gaussian Process).

A stochastic process X is called, *Gaussian* if each of its finite dimensional marginals $(X(t_1), \ldots, X(t_k))$ has a multivariate normal distribution on Euclidean space.

Semimetrics for Gaussian Processes

Theorem.

Let X be a Gaussian process with "intrinsic" semimetrics ρ_p, and let X_n be a sequence of random elements with values in $\ell^\infty(T)$. Then there exists a version of X which is a tight Borel measurable map into $\ell^\infty(T)$ and X_n converges weakly to X if and only if for some p (and then for all p):

(i) The marginals of X_n converge weakly to the corresponding marginals of X

(ii) X_n is asymptotically equicontinuous in probability with respect to ρ_p

(iii) T is totally bounded for ρ_p