Back to Empirical Processes

BIO 251,
Lab 5

March 3, 2014

A Theorem

Theorem.

(i) Let $X_n : \Omega_n \mapsto \ell^\infty(T)$ be arbitrary. Then X_n converges weakly to a tight limit iff X_n is asymptotically tight, and the marginals $(X_n(t_1), \ldots, X_n(t_k))$ converge weakly to a limit for every finite subset $t_1, \ldots, t_k$ of T.

(ii) If X_n is asymptotically tight and it's marginals converge weakly to the marginals $(X(t_1), \ldots, X(t_k))$ of a stochastic process X then there is a version of X with uniformly bounded sample paths and $X_n \rightsquigarrow X$.

Characterization of Asymptotic Tightness

Theorem.

A sequence $X_n : \Omega_n \mapsto \ell^\infty(T)$ is asymptotically tight, if and only if $X_n(t)$ is asymptotically tight in $\mathbb{R}$ for every t, and there exists a semimetric ρ on T such that (T, ρ) is totally bounded and X_n is *asymptotically uniformly ρ-equicontinuous in probability*, i.e. for every $\varepsilon, \eta > 0$, there exist a $\delta > 0$ such that:

$$\limsup_n P^* \left(\sup_{\rho(s,t)<\delta} |X_n(s) - X_n(t)| > \varepsilon \right) < \eta$$

Characterization of Asymptotic Tightness

Addendum.

If, moreover, $X_n \rightsquigarrow X$, then almost all paths $t \mapsto X(t, \omega)$ are uniformly ρ-continuous; and the semimetric ρ can WLOG be taken equal to any semimetric ρ for which this is true and (T, ρ) is totally bounded.

Examples of Semimetrics

Example

- $\rho_0(s,t) = \mathrm{E} \arctan |X(s) - X(t)|$
- $\rho_p(s,t) = [\mathrm{E}\, |X(s) - X(t)|^p]^{1/(p \vee 1)}$

Gaussian Processes

Definition (Gaussian Process).

A stochastic process X is called, *Gaussian* if each of its finite dimensional marginals $(X(t_1), \ldots, X(t_k))$ has a multivariate normal distribution on Euclidean space.

Semimetrics for Gaussian Processes

Theorem.

Let X be a Gaussian process with "intrinsic" semimetrics ρ_p, and let X_n be a sequence of random elements with values in $\ell^\infty(T)$. Then there exists a version of X which is a tight Borel measurable map into $\ell^\infty(T)$ and X_n converges weakly to X if and only if for some p (and then for all p):

(i) The marginals of X_n converge weakly to the corresponding marginals of X

(ii) X_n is asymptotically equicontinuous in probability with respect to ρ_p

(iii) T is totally bounded for ρ_p

Notation Revision

Notation Revision

$$\mathbb{P}_n = n^{-1}\sum_{i=1}^{n}\delta_{X_i}$$

$$\mathcal{F} = f : \mathcal{X} \mapsto \mathbb{R} : f \text{ measurable}$$

$$f \mapsto \mathbb{P}_n f = n^{-1}\sum_{i=1}^{n} f(X_i), \text{ with } Qf = \int f dQ$$

Notation Revision

Notation Revision

$$f \mapsto \mathbb{G}_n f = \sqrt{n}(\mathbb{P}_n f - Pf) = \sqrt{n}\frac{\sum_{i=1}^n f(X_i) - Pf}{n}$$

Interested in $\mathcal{F}$ such that

- $||\mathbb{P}_n - P||_{\mathcal{F}} = \sup_{f\in\mathcal{F}} ||(\mathbb{P}_n - P)f|| \to 0$, outer almost surely
- $\mathbb{G}_n \rightsquigarrow \mathbb{G}$, in $\ell^\infty(\mathcal{F})$

Notation Revision

We have

$$(\mathbb{G}_n f_1, \ldots, \mathbb{G}_n f_k) \rightsquigarrow N_k(0, \Sigma)$$
$$\Sigma_{ij} = P(f_i - Pf_i)(f_j - Pf_j)$$

Thus

The limiting process $\mathbb{G}$ ought to be a Gaussian process, $\{\mathbb{G}f : f \in \mathcal{F}\}$ with zero mean and covariance function:

$$\mathrm{E}\,\mathbb{G}f_1\mathbb{G}f_2 = P(f_1 - Pf_1)(f_2 - Pf_2) = Pf_1 f_2 - Pf_1 Pf_2$$

Orlicz norm

Definition.

Orlicz norm of a non-decreasing, convex function ψ with $\psi(0) = 0$ for a random variable X: $||X||_\psi$ is defined as:

$$||X||_\psi = \inf\{C > 0 : \mathrm{E}\,\psi\left(\frac{|X|}{C}\right) \leq 1\}$$

Examples

Example

- $\psi(x) = x^p$, gives us L_p
- $\psi_p(x) = exp(x^p) - 1$

Some Inequalities

- $||X||_p \leq ||X||_{\psi_p}$
- $||X||_{\psi_p} \leq (\log 2)^{p/q} ||X||_{\psi_q}$, $p \leq q$
- $||X||_p \leq p! ||X||_{\psi_1}$

Tail Bounds with Orlicz norms

Tail Bounds

- $P(|X| > x) \leq \dfrac{1}{\psi(x/||X||_\psi)}$, by Markov's Inequality
- Thus for ψ_p we get a bound of the sort $\approx \exp(-Cx^p)$

Lemma.

Let X be a random variable with $P(|X| > x) \leq K\exp(-Cx^p)$, for every x for some constants K and C, and for $p \geq 1$. Then its Orlicz norm satisfies $||X||_{\psi_p} \leq ((1+K)/C)^{1/p}$.