Weak Convergence and Spaces of Bounded Funcitons

BIO 251,
Lab 2

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Weak Convegence

The “weak convergence of laws without laws being defined” – except asymptotically.

Tightness

Tightness

Definition. A Borel probability measure L is *tight* if for every $\varepsilon > 0$, there exists a compact set K, such that: $L(K) \geq 1 - \varepsilon$.

Weak Convergence

Weak Convergence

Definition. Let $(\Omega_n, \mathcal{A}_n, P_n)$ are sets of probability spaces and $X_n : \Omega_n \mapsto \mathbb{D}$ are arbitrary maps. The sequence X_n converges weakly to a Borel measure L if:

$$\mathrm{E}^* f(X_n) \to \int f dL, \text{ for all } f \in C_b(\mathbb{D})$$

We denote weak convergence by $X_n \rightsquigarrow L$.

Weak Convergence

Theorem (Portmanteau).

TFAE:

(i) $X_n \rightsquigarrow L$

(ii) $\liminf P_*(X_n \in G) \geq L(G)$, for every open G

(iii) $\limsup P^*(X_n \in F) \leq L(F)$, for every closed F

(iv) $\liminf \mathrm{E}_* f(X_n) \geq \int f dL$ for every lower semincontinuous, bounded from below f

(v) $\limsup \mathrm{E}^* f(X_n) \leq \int f dL$ for every upper semicontinuous, bounded from above f

(vi) $\lim P^*(X_n \in B) = \lim P_*(X_n \in B) = L(B)$ for every Borel set B with $L(\delta B) = 0$.

(vii) $\liminf \mathrm{E}_* f(X_n) \geq \int f dL$ for every bounded, Lipschitz continuous, nonnegative f.

Continuous Mapping Theorem

Theorem (Continuous Mapping).

Let $g : \mathbb{D} \mapsto \mathbb{E}$, is a continuous at every point of a set $\mathbb{D}_0 \subset \mathbb{D}$. If $X_n \rightsquigarrow X$ and X takes values in $\mathbb{D}_0$, we have $g(X_n) \rightsquigarrow g(X)$.

Asymptotic Measurability and Tightness

Asymptotic Measurability

Definition. The sequence of maps X_n is *asymptotically measurable* iff

$$\mathrm{E}^* f(X_n) - \mathrm{E}_* f(X_n) \to 0, \text{ for every } f \in C_b(\mathbb{D})$$

Asymptotic Tightness

Definition. The sequence X_n is *asymptotically tight* if for every $\varepsilon > 0$ there exists a compact set K such thats:

$$\liminf P_*(X_n \in K^\delta) \geq 1 - \varepsilon \text{ for every } \delta > 0$$

Where $K^\delta = \{y \in \mathbb{D} : d(y, K) < \delta\}$ is the δ-enlargement around K

Prohorov's Theorem

Lemma.

(i) If $X_n \rightsquigarrow X$, then X_n is asymptotically measurable

(ii) If $X_n \rightsquigarrow X$, then X is tight iff X_n is asymptotically tight.

Theorem (Prohorov).

If the sequence X_n is asymptotically tight and asymptotically measurable, there exists a subsequence X_{n_j} which converges weakly to a tight Borel law.

Is weak convergence different in a subspace?

Theorem.

Let $\mathbb{D}_0 \subset \mathbb{D}$ be arbitrary and let X and X_n take values in $\mathbb{D}_0$. Then $X_n \rightsquigarrow X$ as maps in $\mathbb{D}_0$ iff $X_n \rightsquigarrow X$ as maps in $\mathbb{D}$. Here $\mathbb{D}_0$ and $\mathbb{D}$ are equipped with the same metric.

Is weak convergence to a unique law?

Lemma.

(i) Let L_1 and L_2 are finite Borel measures on $\mathbb{D}$. If $\int f dL_1 = \int f dL_2$ for all $f \in C_b(\mathbb{D})$ then $L_1 \equiv L_2$.

(ii) Let L_1 and L_2 be tight Borel probability measures on $\mathbb{D}$. If $\int f dL_1 = \int f dL_2$ for every f in a vector lattice $\mathcal{F} \subset C_b(\mathbb{D})$ that contains the constant functions, and separates the points in $\mathbb{D}$, then $L_1 \equiv L_2$.

Here vector lattice means a vector space of functions $\mathcal{F}$ such that $f \in \mathcal{F}$ it follows that $f^+ = f \vee 0 \in \mathcal{F}$. $\mathcal{F}$ is said to separate the points if for every two points $x \neq y$ there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.

How to show asymptotic measurability

Lemma.

Let X_n be asymptotically tight, and suppose that $\mathrm{E}^* f(X_n) - \mathrm{E}_* f(X_n) \to 0$ holds for $f \in \mathcal{F}$, where $\mathcal{F}$ is a subalgebra of $C_b(\mathbb{D})$, which separates the points of $\mathbb{D}$. Then X_n is asymptotically measurable.

A subalgebra is a vector space that in addition is closed under taking products.

Space of Bounded Functions

- Let T be an arbitrary set.
- The space $\ell^\infty(T)$ is defined as the set of all uniformly bounded real functions $z : T \mapsto \mathbb{R}$ with:

$$||z||_T := \sup_{t \in T} |z(t)| < \infty$$

Stochastic Process

Stochastic Process

Definition. A *stochastic process* is an indexed collection of random variables $\{X(t) : t \in T\}$ defined on the same probability space:

- i.e. every $X(t) : \Omega \mapsto \mathbb{R}$ is a measurable map.
- If the *sample paths* $t \mapsto X(t, \omega)$ are bounded, then a stochastic process yields a map $X : \Omega \mapsto \ell^\infty(T)$.

Two Lemmas

Lemma 1.

Let $X_n : \Omega_n \mapsto \ell^\infty(T)$ be asymptotically tight. Then it is asymptotically measurable iff $X_n(t)$ is asymptotically measurable for every $t \in T$.

Lemma 2.

Let X and Y be tight Borel measurable maps into $\ell^\infty(T)$. Then X and Y are equal in Borel law iff all corresponding marginals of X and Y are equal in law.

And a Theorem

Theorem.

(i) Let $X_n : \Omega_n \mapsto \ell^\infty(T)$ be arbitrary. Then X_n converges weakly to a tight limit iff X_n is asymptotically tight, and the marginals $(X_n(t_1), \ldots, X_n(t_k))$ converge weakly to a limit for every finite subset $t_1, \ldots, t_k$ of T.

(ii) If X_n is asymptotically tight and it's marginals converge weakly to the marginals $(X(t_1), \ldots, X(t_k))$ of a stochastic process X then there is a version of X with uniformly bounded sample paths and $X_n \rightsquigarrow X$.