Spaces of Bounded Functions

Lab 4, BIO251

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1 Spaces of Bounded Functions

Let T be an arbitrary set. The space $\ell^\infty(T)$ is defined as the set of all uniformly bounded real functions $z : T \mapsto \mathbb{R}$ with:

$$||z||_T := \sup_{t \in T} |z(t)| < \infty$$

Definition. A *stochastic process* is simply an indexed collection of random variables $\{X(t) : t \in T\}$ defined on the same probability space: every $X(t) : \Omega \mapsto \mathbb{R}$ is a measurable map. If the *sample paths* $t \mapsto X(t, \omega)$ are bounded, then a stochastic process yields a map $X : \Omega \mapsto \ell^\infty(T)$.

The amount of measurability given by the stochastic process definition, might already be enough for asymptotic measurability. The *marginals* $(X(t_1), X(t_2), \dots, X(t_k))$ play a special role for weak convergence, when considered as maps into $\mathbb{R}^k$, and this is obvious after the following three lemmas.

Lemma 1. Let $X_n : \Omega_n \mapsto \ell^\infty(T)$ be asymptotically tight. Then it is asymptotically measurable iff $X_n(t)$ is asymptotically measurable for every $t \in T$.

Lemma 2. Let X and Y be tight Borel measurable maps into $\ell^\infty(T)$. Then X and Y are equal in Borel law iff all corresponding marginals of X and Y are equal in law.

Theorem. Let $X_n : \Omega_n \mapsto \ell^\infty(T)$ be arbitrary. Then X_n converges weakly to a tight limit iff X_n is asymptotically tight, and the marginals $(X_n(t_1), \dots, X_n(t_k))$ converge weakly to a limit for every finite subset $t_1, \dots, t_k$ of T. If X_n is asymptotically tight and it's marginals converge weakly to the marginals $(X(t_1), \dots, X(t_k))$ of a stochastic process X then there is a version of X with uniformly bounded sample paths and $X_n \rightsquigarrow X$.

Proofs. For the first two lemmas consider the collections of functions $\mathcal{F}$, s.t. for $f \in \mathcal{F}$ we have $f : \ell^\infty(T) \mapsto \mathbb{R}$ of the form:

$$f(z) = g(z(t_1), \dots, z(t_k)), g \in C_b(\mathbb{R}^k), t_i \in T, k \in \mathbb{N}$$

Obviously, $\mathcal{F}$ forms a vector lattice and a subalgebra, and contains the constant functions, since the functions g would stay bounded and continuous under multiplication or taking the positive parts. Furthermore, it also separates the points as if two functions $z_1 \neq z_2$ differ at time $t \in T$, then the projection $f(z) = z(t)$ (note that this is a continuous function) separates the two points and $f \in \mathcal{F}$.

Lemma 1. Assume X is asymptotically measurable. Then set $f(X_n) = g(X_n(t))$ to be the projection on t and then asymptotic measurability follows by the definition. Conversely assume that $X_n(t)$ is asymptotically measurable for all t. Using the fact that asymptotic measurability of two asymptotically tight random sequences implies their joint asymptotic measurability (not trivial needs proof but we omit it!) we can see that all finite dimensional marginals are asymptotically measurable. Then the last lemma of the weak convergence section and the fact that $\mathcal{F}$ is a subalgebra give us the desired result.

Lemma 2. If $X = Y$, then the marginals ought to be the same. If the marginals are the same the fact that $\mathcal{F}$ is a lattice containing constants, and separating the points, we get that $X = Y$ from part (ii) of the next to last lemma of the weak convergence section.

Theorem Proof. If X_n is asymptotically tight and the marginals converge weakly to a tight limit, then by Lemma 1 X_n is asymptotically measurable. Now we can apply Prohorov's theorem to get that we can divide X_n into subsequences each of which is converging weakly to a tight limit. If we show that each limit is the same we are done. Because we have the marginal convergence (on all subsequences), and Lemma 2 this finishes the proof in this direction. Now if X_n converges weakly to a tight limit we know that this is equivalent to X_n being asymptotically tight. Furthermore the marginals weak convergence comes as an implication of the continuous mapping theorem (note that the map $f(z) = (z(t_1), \dots, z(t_k))$ is continuous for $z \in \ell^\infty(T)$).

Finally note that if X_n is asymptotically tight and the marginals converge, by the first part we have that $X_n \rightsquigarrow X$, where X is tight limit. Furthermore, since X is tight, then it concentrates on a σ-compact set $K \subset \ell^\infty(T)$ with $P(X \in K) = 1$. The last implication shows the uniform boundedness of the sample paths.

2 Characterizing Asymptotic Tightness in $\ell^\infty(T)$

While we know how to deal with the marginal weak convergence, characterizations of the asymptotic tightness are needed, in order to be able to prove weak convergence of the empirical process. We provide certain characterizations of asymptotic tightness below.

We formulate a theorem which is essentially relating the (asymptotic uniform, equi-) continuity of the sample paths $t \mapsto X_n(t)$ to asymptotic tightness.

We have the following theorem:

Theorem. A sequence $X_n : \Omega_n \mapsto \ell^\infty(T)$ is asymptotically tight, if and only if $X_n(t)$ is asymptotically tight in $\mathbb{R}$ for every t, and there exists a semimetric ρ on T such that (T, ρ) is totally bounded and X_n is *asymptotically uniformly ρ-equicontinuous in probability*, i.e. for every $\varepsilon, \eta > 0$, there exist a $\delta > 0$ such that:

$$\limsup_n P^* \left(\sup_{\rho(s,t) < \delta} |X_n(s) - X_n(t)| > \varepsilon \right) < \eta$$

Futhermore we have:

Addendum. If, moreover, $X_n \rightsquigarrow X$, then almost all paths $t \mapsto X(t, \omega)$ are uniformly ρ-continuous; and the semimetric ρ can WLOG be taken equal to any semimetric ρ for which this is true and (T, ρ) is totally bounded.

Let's spend some time on understanding the theorem and the addendum.

It is not surprising that we require asymptotic tightness for each of the coordinate projections $X_n(t)$, since any continuous function $g(X_n)$ should be asymptotically tight if X_n is asymptotically tight, and the projection is a particular example of that. This is true because a continuous image of a compact set is compact.

What does a totally bounded set mean? Totally bounded set, is a set such that for each $\varepsilon > 0$ we can find a finite cover of the set consisting of open balls of radius ε of the set. So the theorem is really saying that we can "chop up" the space T into finite number of balls of radius δ, and on each of these balls, for all n it is very likely that our X_n's restricted to a ball should be very close to each other with high probability.

In other words if we denote the balls with B_i, $i = 1, \dots, k$, we have that the behavior of the processes $X_n(t)$ could be explained by the marginal distributions $(X_n(t_1), \dots, X_n(t_k))$, where $t_i \in B_i$ up to errors of ε, η.

The addendum, seems also to make intuitive sense. If the sample paths of the sequence of processes become to act similarly on the balls B_i then so must be true for the sample paths of X, hence continuity should be expected. Furthermore, if many metrics ρ are making (T, ρ) totally bounded and the sample paths of X uniformly ρ-continuous it would be "unnatural" if we could only use some of these, to show the asymptotic uniform equicontinuity. Fortunately the addendum guarantees that that's not the case.

There is a question of what semimetrics one should try in practice. Examples could be $\rho_0(s,t) = \mathrm{E} \arctan |X(s) - X(t)|$, $\rho_p(s,t) = (\mathrm{E} |X(s) - X(t)|^p)^{1/(p \vee 1)}$, $0 < p < \infty$. Checking that these are semimetrics is left as an exercise (Hints: consider the function $\arctan(x) + \arctan(c - x)$, for $0 \leq x \leq c$, $|a+b|^p \leq |a|^p + |b|^p$ for $p < 1$, and Minkowski's inequality).

It can be shown that ρ_0, would do the job, however it might not be convenient to use. For the ρ_p metrics it is not clear wether they would work, as the expectations need not even be finite.

Turns out however, that for the Gaussian process, we shouldn't be worried about using ρ_p.

2.1 Gaussian Processes

Def. A stochastic process X is called, *Gaussian* if each of its finite dimensional marginals $(X(t_1), \dots, X(t_k))$ has a multivariate normal distribution on Euclidean space.

Theorem. Let X be a Gaussian process with "intrinsic" semimetrics ρ_p, and let X_n be a sequence of random elements with values in $\ell^\infty(T)$. Then there exists a version of X which is a tight Borel measurable map into $\ell^\infty(T)$ and X_n converges weakly to X if and only if for some p (and then for all p):

(i) The marginals of X_n converge weakly to the corresponding marginals of X

(ii) X_n is asymptotically equicontinuous in probability with respect to ρ_p

(iii) T is totally bounded for ρ_p

Typically one uses ρ_2 as a semi metric as it is the easiest to work with.

The sufficiency of these conditions should be evident from the theorem above. However in the Gaussian process case, we have also that these conditions are necessary, i.e. we can always use any of the semimetrics ρ_p. This is not obvious form the theorem but we don't discuss it further here.