VC Classes

BIO 251,
Lab 10

April 28, 2014

Some Definitions

- Let $\mathcal{C}$ be a collection of subsets of a set $\mathcal{X}$.
- Say that $\mathcal{C}$, *picks out* a certain subset from $\{x_1, \dots, x_n\}$ if this subset takes the form $C \cap \{x_1, \dots, x_n\}$ for some $C \in \mathcal{C}$.
- $\mathcal{C}$ is said to *shatter* $\{x_1, \dots, x_n\}$ if all possible 2^n subsets can be picked out by $\mathcal{C}$.

VC-index

The *VC-index* $V(\mathcal{C})$ of the collection $\mathcal{C}$ is the smallest n for which there is no set of size n, which is shattered by $\mathcal{C}$.

VC-Index

$$\Delta_n(\mathcal{C}, x_1, \ldots, x_n) = \#\left\{C \cap \{x_1, \ldots, x_n\} : C \in \mathcal{C}\right\}$$
$$V(\mathcal{C}) = \inf\{n : \max_{x_1,\ldots,x_n} \Delta_n(\mathcal{C}, x_1, \ldots, x_n) < 2^n\}$$

Example

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- The collections of all cells of the form $(-\infty, c]$
- The collection of sets $(a, b]$ for $a, b \in \mathbb{R}$

A Combinatorial Lemma

Lemma (Sauer-Shelah).

Let $\{x_1, \ldots, x_n\}$ be arbitrary points. Then the total number of subsets $\Delta_n(\mathcal{C}, x_1, \ldots, x_n)$ picked out by $\mathcal{C}$ is bounded above by the number of subsets of $\{x_1, \ldots, x_n\}$ shattered by $\mathcal{C}$.

Corollary.

For a VC-class of sets of index $V(\mathcal{C})$, one has:

$$\max_{x_1, \ldots, x_n} \Delta_n(\mathcal{C}, x_1, \ldots, x_n) \leq \sum_{j=0}^{V(\mathcal{C})-1} \binom{n}{j}$$

Theorem

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There exists a universal constant K such that for any VC-class $\mathcal{C}$ of sets, any probability measure Q, any $r \geq 1$, and $0 < \varepsilon < 1$, we have:

$$N(\varepsilon, \mathcal{C}, L_r(Q)) \leq KV(\mathcal{C})(4e)^{V(\mathcal{C})} \left(\frac{1}{\varepsilon}\right)^{r(V(\mathcal{C})-1)}$$

A Weaker Result

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For every $\delta > 0$ there exists a constant K, depending on $V(\mathcal{C})$ and δ only, such that for any VC-class $\mathcal{C}$ of sets, any probability measure Q, any $r \geq 1$, and $0 < \varepsilon < 1$, we have:

$$N(\varepsilon, \mathcal{C}, L_r(Q)) \leq K \left(\frac{1}{\varepsilon}\right)^{r(V(\mathcal{C})-1+\delta)}$$